ISRAEL JOURNAL OF MATHEMATICS 123 (2001), 211–220

A PATHOLOGICAL EXAMPLE OF A UNIFORM QUOTIENT MAPPING BETWEEN EUCLIDEAN SPACES

BY

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ABSTRACT

A uniform quotient Lipschitz mapping between Euclidean spaces of dimensions n and $n-1$, which annihilates the unit ball of a hyperplane, is constructed.

1. Introduction

This work is inspired by the paper [BJLPS], where Lipschitz quotient mappings and uniform quotient mappings are studied. A map $f\colon X \to Y$, where X and Y are metric spaces, is called a **uniform quotient** if

$$B_{\Omega(r)}(f(x)) \supset f(B_r(x)) \supset B_{\omega(r)}(f(x))$$

for any $x \in X$ and $r > 0$, where $\omega(r)$ and $\Omega(r)$ are functions of the radius r independent of the point x, such that $\omega(r) > 0$ for $r > 0$ and $\Omega(r) \to 0$ as $r \downarrow 0$. If the first inclusion holds, f is called **uniformly continuous**; if the second holds, f is called **co-uniformly continuous** or **co-uniform.** If $\omega(r) \geq cr$ and $\Omega(r) \leq Cr$ for some $c, C > 0$, f is said to be a **Lipschitz quotient mapping** (**co-Lipschitz** if the first inequality holds and **Lipschitz** if the second inequality holds).

There is a developed theory of uniform/Lipschitz quotient mappings which are one-to-one ([BL]), but not much is known in the general case.

* Supported by the Israel Science Foundation.
Received July 1, 1999 and in revised form April 6, 2000

For example, if X and Y are Banach spaces, then the Gorelik principle ([G], [JLS]) says that one-to-one uniform quotient mapping cannot carry the unit ball in a finite codimensional subspace of X into a "small" neighborhood of an infinite codimensional subspace of Y. The proof of the Gorelik principle actually shows that a bi-uniform homeomorphism cannot map a ball in a subspace of codimension k into a small neighborhood of a subspace of codimension $k+1$. This holds regardless of whether X and Y are finite or infinite dimensional.

One may ask whether a similar principle holds for uniform quotient mappings, which are not one-to-one. It turns out that this is not the case even for finite dimensional spaces.

As was proved in [BJLPS], for each n there is a uniform quotient mapping from $\mathbb{R}^{2n+1}$ onto $\mathbb{R}^n$ which maps the unit ball of the hyperplane to zero. Moreover, there is a stronger example for low dimensions: A Lipschitz and co-uniform mapping from $\mathbb{R}^3$ onto $\mathbb{R}^2$ which annihilates the unit ball of a hyperplane.

In the present paper we generalize this construction to the case of arbitrary dimension. The result of the paper reads as follows:

For $n \geq 1$ there is a Lipschitz and co-uniform mapping T from $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \oplus \mathbb{R}$ onto $\mathbb{R}^{n+1}$ such that $T(B_1^{\mathbb{R}^{n+1}\oplus 0}(0)) = \{0\}$.

2. The idea of the construction

Before going into the technical details we briefly describe the example and the proof in an informal way. The space $\mathbb{R}^{n+2}$ is decomposed into the direct sum $\mathbb{R}^{n+1} \oplus \mathbb{R} = \{(x, a) \mid x \in \mathbb{R}^{n+1}, a \in \mathbb{R}\}$, and the mapping is of the form $T(x, a) = \varphi_a(\|x\|) \cdot U_{\psi_a(\|x\|)} x$, where $U_{(\cdot)}$ is a family of orthogonal operators acting on $\mathbb{R}^{n+1}$. This family together with the functions $\varphi_a(\|x\|)$ and $\psi_a(\|x\|)$ are chosen in such a way that the mapping T is clearly Lipschitz.

The main part of the proof deals with the co-uniformity of T, namely we check the inclusion $TB_r(x, a) \supset B_{\omega(r)}(T(x, a))$ for a fixed radius $r > 0$. It turns out that if a or $\|x\|$ is large enough, more exactly if $\|x\| > 1 + \alpha_1 r^n$ or if $|a| > \alpha_2 r$ for suitably chosen constants α_1 and α_2, then for a fixed and y close to $f_a(x) = T(x, a)$ in $\mathbb{R}^n$, the gradient of $f_a^{-1}(y)$ is uniformly bounded in norm by a certain constant c, depending on r. So $TB_r(x, a) \supset T(B_r(x), a) \supset B_{r/c}(T(x, a))$.

The other case is: $\|x\|$ is less than 1 (or not much greater than 1) and $|a| \leq \alpha_2 r$. In this case the inclusion $TB_r(x, a) \supset B_{\omega(r)}(T(x, a))$ is of different nature. If x remains fixed and a runs over $[0, \alpha_2 r]$ (so the point (x, a) does not leave the ball of radius r), the point $T(x, a)$ "draws" a curve which is "dense" in the ball $B_{\|x\|c(r)}(0)$ in the sense that its small neighborhood contains

$B_{\|x\|c(r)}(0) \supset B_{\omega(r)}(T(x,a))$. This small neighborhood is contained, say, in the image of $B_{r/2}(x) \times [\,0, \alpha_2 r\,] \subset B_r(x,a)$, so the inclusion follows. This remarkable Lipschitz curve $T(x, [\,0, \alpha_2 r\,])$ looks like a spiral of infinitely many turns around 0, when $x \in \mathbb{R}^2$ (see Fig. 1 below). In higher dimensions the curve is some spatial analogue of such a spiral.

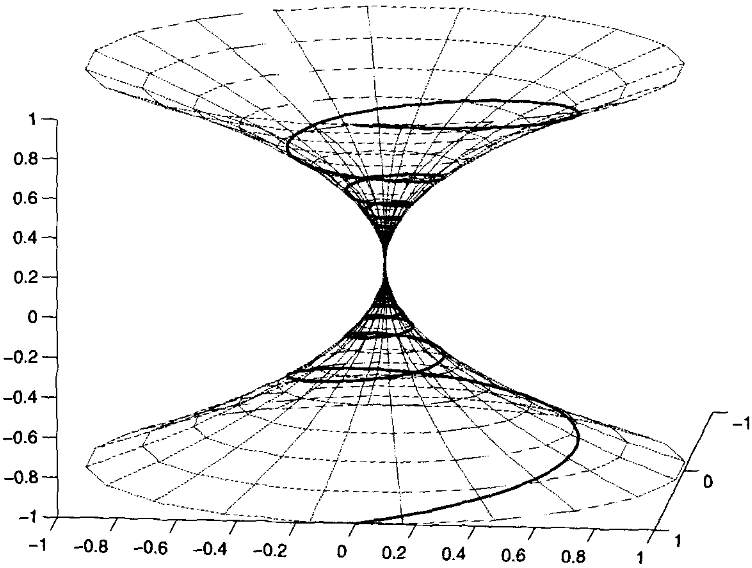

Figure 1. The image $T((0,1),a)$, $-1 \leq a \leq 1$ is the projection of the bold curve onto the bottom plane.

In this part we use a special lemma, which allows us to approximate a fixed finite sequence of angles by residues of $2\pi/\gamma$, $2\pi/\gamma^2, \ldots, 2\pi/\gamma^n$ modulo 2π.

The question, whether there exists a Lipschitz quotient mapping from $\mathbb{R}^n$ onto $\mathbb{R}^m$ which annihilates an object of dimension greater than $n - m$, remains open.

ACKNOWLEDGEMENT: This work will form part of a Ph.D. thesis written under the supervision of Professor Gideon Schechtman. The author thanks Professor Schechtman for suggesting the problem and for useful discussions.

3. The construction

THEOREM 3: *For $n \geq 1$ there is a Lipschitz mapping T from $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \oplus \mathbb{R}$ onto $\mathbb{R}^{n+1}$ such that T is a co-uniform quotient mapping and $T(B_1^{\mathbb{R}^{n+1}\oplus 0}(0)) = \{0\}$.*

Proof: Let x_k be the kth coordinate vector of the space $\mathbb{R}^{n+1}$, and Ox_kx_{k+1} denote the coordinate plane spanned by x_k, x_{k+1}. We interpret $\mathbb{R}^k$ as the subspace of $\mathbb{R}^{n+1}$ spanned by $x_1, \ldots, x_k$. Denote by π_k the standard orthogonal projection $\mathbb{R}^{n+1} \to \mathbb{R}^k$. Let S_r^k denote a sphere in $\mathbb{R}^{k+1}$ of radius r, centered at zero. By $R^{\alpha}_{Ox_kx_{k+1}}$ we mean the orthogonal transformation of the space, which acts as planar rotation by α in the kth and $(k+1)$th coordinates, leaving the rest of the coordinates unchanged. Note that

$$\text{(1)} \qquad \begin{gathered} \text{if } \|v\| = \|w\| \text{ and } v - w \in Ox_kx_{k+1}, \\ \text{then } w = R^{\alpha}_{Ox_kx_{k+1}} v \text{ for some } \alpha \in [0, 2\pi]. \end{gathered}$$

We define the orthogonal operator $U^{[k+1]}_{\alpha_1,\ldots,\alpha_k}$ inductively by

$$\begin{aligned} U^{[2]}_{\alpha} &= R^{\alpha}_{Ox_1x_2}, \\ U^{[k+1]}_{\alpha_1,\ldots,\alpha_k} &= (U^{[k]}_{\alpha_2,\ldots,\alpha_k})^{-1} R^{\alpha_1}_{Ox_kx_{k+1}} U^{[k]}_{\alpha_2,\ldots,\alpha_k}. \end{aligned}$$

For x fixed and α_j running over $[0, 2\pi]$ independently, $U^{[n+1]}_{\alpha_1,\ldots,\alpha_n}(x)$ runs over the whole sphere in $\mathbb{R}^{n+1}$ of radius $\|x\|$, centered at the origin.

To show this, let us note first that $\{U^{[2]}_{\alpha}(x) \mid \alpha \in [0, 2\pi]\} = S^1_{\|x\|}$ for $x \in \mathbb{R}^2$. Assume that $U^{[k]}_{\alpha_1,\ldots,\alpha_{k-1}}(x)$ runs over the whole sphere $S^{k-1}_{\|x\|}$ for fixed $x \in \mathbb{R}^k$. Now fix $x \in \mathbb{R}^{k+1}$ and take arbitrary $y \in S^k_{\|x\|}$. Since $\pi_k(x - y) \in \mathbb{R}^k$, there exist $\alpha_2, \ldots, \alpha_k$ such that $U^{[k]}_{\alpha_2,\ldots,\alpha_k}\pi_k(x - y) = \pi_k U^{[k]}_{\alpha_2,\ldots,\alpha_k}(x - y) = \|\pi_k(x - y)\| x_k$. Then $U^{[k]}_{\alpha_2,\ldots,\alpha_k}(x - y)$ lies in Ox_kx_{k+1}. By (1), there exists α_1 such that

$$U^{[k]}_{\alpha_2,\ldots,\alpha_k} y = R^{\alpha_1}_{Ox_kx_{k+1}} U^{[k]}_{\alpha_2,\ldots,\alpha_k} x.$$

By definition this means that $U^{[k+1]}_{\alpha_1,\ldots,\alpha_k} x = y$.

For $u \in \mathbb{R}$, let $d_u \colon \mathbb{R}_+ \to [0, 1]$ be the continuous function such that $d_u(t) = \min(|u|, 1)$ for $t \le 1$, $d_u(t) = 1$ for $t \ge 2$, $d_u(t)$ is linear for $1 \le t \le 2$.

Define $T \colon \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ by

$$T(x, a) = d^2_{a^n}(\|x\|) U^{[n+1]}_{2\pi/d_a(\|x\|), 2\pi/d_{a^2}(\|x\|), \ldots, 2\pi/d_{a^n}(\|x\|)} x.$$

Note that for $n = 1$ this reduces to the construction in [BJLPS].

Let us check that T is a Lipschitz mapping. For $\|x\| \ge 2$ this is clear, since $T(x, a) = x$. The restriction of T to the set $\{(x, a) \colon \|x\| \le 2\}$ is the composition of a Lipschitz mapping

$$(x, a) \mapsto (x, d_a(\|x\|), d_{a^2}(\|x\|), \ldots, d_{a^n}(\|x\|)),$$

with

$$(x, t_1, \ldots, t_n) \in \{(x, t_1, \ldots, t_n) : \|x\| \leq 2, 0 \leq t_n \leq \cdots \leq t_1 \leq 1\} \mapsto t_n^2 U^{[n+1]}_{2\pi/t_1, \ldots, 2\pi/t_n} x;$$

the latter is 1-Lipschitz in x, and each entry of the matrix

$$t_n^2 U^{[n+1]}_{2\pi/t_1, \ldots, 2\pi/t_n}$$

is a combination of $\sin(2\pi/t_i)$ and $\cos(2\pi/t_i)$, multiplied by t_n^2; as $t_n^2 \leq t_i^2$, such an expression has bounded partial derivatives in t_i.

Let us begin the proof of the co-uniformity of T with the following Lemma.

Lemma 1: *For $0 < \rho < 1$ there exists a constant c_ρ depending only on ρ and n, such that*

$$T(B_\rho(x), a) \supset B_{c_\rho}(T(x, a)), \quad \textit{if either } a^n > \rho \textit{ or } \|x\| > 1 + \rho.$$

Proof: Note that for each nonzero a the inverse of the mapping

$$f_a(x) = T(x, a) \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

can be obtained as

$$f_a^{-1}(y) = \frac{p_a(\|y\|)}{\|y\|} \left(U^{[n+1]}_{2\pi/d_a(p_a(\|y\|)), \ldots, 2\pi/d_{a^n}(p_a(\|y\|))}\right)^{-1} y,$$

where $p_a(t)$ is the inverse of $q_a(t) = t d_{a^n}^2(t)$ (the above holds also for $a = 0$ as long as $\|x\| > 1$). For $t \in (0,1) \cup (1,2) \cup (2, \infty)$, the derivative of $q_a(t)$ is bounded below by $d_{a^n}^2(t)$, i.e., is not less than $a^{2n} \wedge 1$; moreover, $d_{a^n}^2(t)$ is bounded below by ρ^2, when $t > 1 + \rho$. Thus, if either $a^n \geq \rho > 0$ or $\|x\| \geq 1 + \rho$, the derivative $p_a'(\|y\|)$ is not greater than $1/\rho^2$ for $y = f_a(x)$. Let us compute the ith partial derivative of f_a^{-1} at $y = f_a(x)$; note that $p_a(\|y\|) = \|x\|$:

$$(2) \quad \frac{\partial f_a^{-1}(y)}{\partial y_i} = \frac{p_a'(\|y\|) y_i}{\|y\|^2} U(p_a(\|y\|)) y - \frac{p_a(\|y\|) y_i}{\|y\|^3} U(p_a(\|y\|)) y + \frac{p_a(\|y\|)}{\|y\|} U(p_a(\|y\|)) e_i + \frac{p_a(\|y\|)}{\|y\|} p_a'(\|y\|) \frac{y_i}{\|y\|} \cdot U'(p_a(\|y\|)) y,$$

where $U(t)$ stands for $(U^{[n+1]}_{2\pi/d_a(t), \ldots, 2\pi/d_{a^n}(t)})^{-1}$. The norm of the first summand is less than or equal to $1/\rho^2$, the norm of the second is less than or equal to

$$\frac{p_a(\|y\|)}{\|y\|} = \frac{1}{d_{a^n}^2(\|x\|)} \leq \frac{1}{\rho^2},$$

the norm of the third is less than or equal to

$$\frac{p_a(\|y\|)}{\|y\|} \leq \frac{1}{\rho^2}.$$

If $t = p_a(\|y\|) \geq 2$ then $U'(t) = 0$, therefore the norm of the fourth summand is less than or equal to

$$\frac{2}{\|y\|}\frac{1}{\rho^2}\|U'(t)\|\|y\|.$$

It remains to estimate the norm of the matrix $\|U'(t)\|$. The matrix $(U^{[n+1]}_{\alpha_1,\dots,\alpha_n})^{-1}$ is the product of $2^n - 1$ rotations in 2-dimensional planes by $\pm\alpha_i$; the derivative of such a rotation with respect to α_j is either zero (if $i \neq j$) or an orthogonal matrix, so

$$\left\|\frac{\partial}{\partial\alpha_j}(U^{[n+1]}_{\alpha_1,\dots,\alpha_n})^{-1}\right\| \leq 2^n - 1.$$

Therefore

$$\begin{aligned}\|U'(t)\| &\leq (2^n - 1)\sum_{j=1}^{n}\left|\left(\frac{2\pi}{d_{a^j}(t)}\right)'\right| \leq 2\pi(2^n - 1)\sum_{j=1}^{n}\frac{d'_{a^j}(t)}{d^2_{a^j}(t)} \\ &\leq \frac{2\pi(2^n - 1)n}{d^2_{a^n}(t)} \leq \frac{C}{\rho^2},\end{aligned}$$

as $d_{a^n}(t) \leq d_{a^j}(t)$ and $d'_{a^j}(t) \leq 1$. Thus, the last summand in the right-hand side of (2), as well as the whole gradient of f_a^{-1} at the point $f_a(x)$, has norm not greater than c/ρ^4 for some c depending on n.

We have proved an intermediate result: if either $a^n > \rho$ or the norm $\|p_a(y)\| > 1 + \rho$, then $\|\nabla f_a^{-1}(y)\| \leq c\rho^{-4}$ for some constant $c \geq 1$ depending only on n.

Now in the case $a^n \geq \rho$ the norm of the gradient of $f_a^{-1}(y)$ is bounded by the same constant $c\rho^{-4}$ at all the points y, so the preimage $f_a^{-1}(B_{\rho^5/c}(f_a(x)))$ is contained in $B_\rho(x)$, which is equivalent to $T(B_\rho(x), a) \supset B_{\rho^5/c}(T(x, a))$.

Let us examine the other case: $\|x\| \geq 1 + \rho$. Note that

$$q_a(\|x\|) - q_a\Big(1 + \frac{\rho}{2}\Big) \geq \frac{\rho}{2}\min_{\xi\geq 1+\rho/2} q'_a(\xi) \geq \Big(\frac{\rho}{2}\Big)^3.$$

Therefore for all $z \in B_{\rho^5/16c}(f_a(x))$ we have

$$\|z\| \geq \|f_a(x)\| - \rho^5/16c \geq \|f_a(x)\| - \rho^3/8 \geq q_a\Big(1 + \frac{\rho}{2}\Big),$$

so the norm of the gradient of f_a^{-1} at z is bounded above by $16c/\rho^4$, as $p_a(\|z\|) \geq 1 + \rho/2$. This means that $f_a^{-1}(B_{\rho^5/16c}(f_a(x))) \subset B_\rho(x)$, which is equivalent to $T(B_\rho(x), a) \supset B_{\rho^5/16c}(T(x, a))$. ∎

Now let us show that T is co-uniform. We may consider only (x, a) in $\mathbb{R}^{n+2}$ with $a \geq 0$ and assume that the radius r lies between 0 and 1.

FIRST CASE. $r \leq 2^{n+9}a$ or $\|x\| > 1 + (r/2^{n+9})^n$. Let $\rho = (r/2^{n+9})^n$; then Lemma 1 implies that

$$TB_r(x, a) \supset T(B_\rho(x), a) \supset B_{c_\rho}(T(x, a)).$$

SECOND CASE. $r > 2^{n+9}a$ and $\|x\| \leq 1$. Let us show that the set

$$\left\{T\Big(\frac{c}{\gamma^{2n}}y, \gamma\Big) \Big| \frac{1}{k+1} \leq \gamma \leq \frac{1}{k}, \|y\| = \|x\|, \|y - x\| \leq \frac{r}{4}\right\}$$

coincides with the sphere $S_{c\|x\|}$ of radius $c\|x\|$, centered at zero, whenever $k \geq 2^{n+5}/r$ is an integer and

$$\frac{1}{(k+2)^{2n}} \leq c \leq \frac{1}{(k+1)^{2n}}.$$

Take $z \in \mathbb{R}^{n+1}$ of norm $c\|x\|$. Fix $\varphi_1, \ldots, \varphi_n \in [0, 2\pi]$ such that $U^{[n+1]}_{\varphi_1,\ldots,\varphi_n}x = z/c$. The following lemma will be proved later:

LEMMA 2: *For any $\varphi_1, \varphi_2, \ldots, \varphi_n \in [0, 2\pi]$ and any positive integer $k \geq 2$ there exists $\gamma \in [\frac{1}{k+1}, \frac{1}{k}]$ such that*

$$\|U^{[n+1]}_{\varphi_1,\varphi_2,\ldots,\varphi_n}x - U^{[n+1]}_{2\pi/\gamma, 2\pi/\gamma^2,\ldots,2\pi/\gamma^n}x\| \leq 2^{n+1}\pi/k \tag{3}$$

for all x: $\|x\| \leq 1$.

Now find $\gamma \in [\frac{1}{k+1}, \frac{1}{k}]$ such that (3) holds. Then

$$\frac{z}{c} \in B_{2^{n+1}\pi/k}(U^{[n+1]}_{2\pi/\gamma, 2\pi/\gamma^2,\ldots,2\pi/\gamma^n}(x)) = U^{[n+1]}_{2\pi/\gamma, 2\pi/\gamma^2,\ldots,2\pi/\gamma^n}B_{2^{n+1}\pi/k}(x),$$

i.e., $z/c = U^{[n+1]}_{2\pi/\gamma, 2\pi/\gamma^2,\ldots,2\pi/\gamma^n}(y)$ for some $y \in B_{2^{n+1}\pi/k}(x) \cap S_{\|x\|}$. This means that

$$z = T\Big(\frac{c}{\gamma^{2n}}y, \gamma\Big), \quad \|y\| = \|x\| \quad \text{and} \quad \|y - x\| \leq 2^{n+1}\pi/k \leq r\frac{2^{n+1}\pi}{2^{n+5}} \leq r/4,$$

which proves the statement.

We have

$$\Big\|(x, a) - \Big(\frac{c}{\gamma^{2n}}y, \gamma\Big)\Big\|^2 \leq \Big(\|x - y\| + \Big|1 - \frac{c}{\gamma^{2n}}\Big|\Big)^2 + |a - \gamma|^2.$$

Now let k run over all integers greater than $2^{n+5}/r$. For each k, let γ run over $[\frac{1}{k+1}, \frac{1}{k}]$, c run over $[\frac{1}{(k+2)^{2n}}, \frac{1}{(k+1)^{2n}}]$ and y run over the set $\{y \mid \|y\| = \|x\|, \|y-x\| \leq r/4\}$. For such γ, c and y we have

$$1 - \frac{c}{\gamma^{2n}} = \Big(1 - \frac{\sqrt{c}}{\gamma^n}\Big)\Big(1 + \frac{\sqrt{c}}{\gamma^n}\Big) \leq 2\Big(1 - \frac{\sqrt{c}}{\gamma^n}\Big)$$
$$\leq 2\Big(1 - \frac{k^n}{(k+2)^n}\Big) \leq 4\frac{n(k+2)^{n-1}}{(k+2)^n} \leq \frac{4n}{k} \leq \frac{4n}{2^{n+5}}r;$$

since $0 \leq a < r/2^{n+5}$ and $0 < \gamma \leq 1/k \leq r/2^{n+5}$ we obtain $|a-\gamma|^2 \leq (r/2^{n+5})^2$ and thus

$$\Big(\|x-y\| + \Big|1 - \frac{c}{\gamma^{2n}}\Big|\Big)^2 + |a-\gamma|^2 \leq \Big(\frac{r}{4} + \frac{4rn}{2^{n+5}}\Big)^2 + \Big(\frac{r}{2^{n+5}}\Big)^2 < r^2.$$

This means that all the points $(\frac{c}{\gamma^{2n}}y, \gamma)$ as above lie in the ball $B_r(x,a)$. Consequently,

$$TB_r(x,a) \supset \bigcup_{0\leq c\leq (r/2^{n+6})^{2n}} S_{c\|x\|} = B_{\|x\|r^{2n}/(2^{n+6})^{2n}}(0), \tag{4}$$

as c runs over

$$\Big[0, \Big(\frac{r}{2r+2^{n+5}}\Big)^{2n}\Big] \supset \Big[0, \Big(\frac{r}{2^{n+6}}\Big)^{2n}\Big].$$

Note that formula (4) holds for all x, a, r such that $0 \leq a < r/2^{n+5}$ and $\|x\| \leq 1$.

Since

$$\|T(x,a)\| = a^{2n}\|x\| \leq \Big(\frac{r}{2^{n+9}}\Big)^{2n}\|x\| \leq \frac{r^{2n}\|x\|}{4(2^{n+6})^{2n}},$$

we conclude that

$$TB_r(x,a) \supset B_{\|x\|r^{2n}/(2^{n+9})^{2n}}(T(x,a)).$$

Now if $\|x\| \geq r/2$ then

$$TB_r(x,a) \supset B_{r/2\cdot r^{2n}/(2^{n+9})^{2n}}(T(x,a)) = B_{r^{2n+1}/2(2^{n+9})^{2n}}(T(x,a)),$$

while if $\|x\| < r/2$ then, putting $y = rx/(2\|x\|)$,

$$TB_r(x,a) \supset TB_{r/2}(rx/(2\|x\|),a) = TB_{r/2}(y,a)$$
$$\supset B_{\|y\|(r/2)^{2n}/(2^{n+6})^{2n}}(0) = B_{(r/2)^{2n+1}/(2^{n+6})^{2n}}(0) \supset B_{r^{2n+1}/(2^{2n+2}(2^{n+6})^{2n})}(z)$$

for all $\|z\| \leq r^{2n+1}/(2^{2n+2}(2^{n+6})^{2n})$. Here formula (4) is valid for the triple $y, a, r/2$, since the conditions $0 \leq a < r/2^{n+6}$ and $\|y\| \leq 1$ hold. But

$$\|T(x,a)\| = a^{2n}\|x\| \leq \Big(\frac{r}{2^{n+9}}\Big)^{2n} \cdot r/2 \leq \frac{r^{2n+1}}{2^{2n+2}(2^{n+6})^{2n}},$$

so $TB_r(x,a) \supset B_{r^{2n+1}/(2^{2n+2}(2^{n+6})^{2n})}(T(x,a))$.

THIRD CASE. $r > 2^{n+9}a$ and $1 < \|x\| \le 1 + (r/2^{n+9})^n$. By (4)

$$TB_r(x,a) \supset TB_{r(1-1/(2^{n+9})^n)}\left(\frac{x}{\|x\|}, a\right) \supset B_{(r(1-1/(2^{n+9})^n))^{2n}/(2^{n+6})^{2n}}(0).$$

Now formula (4) is valid since

$$a < r/2^{n+9} < r\left(1 - \frac{1}{(2^{n+9})^n}\right)/2^{n+5}.$$

Since

$$d_{a^n}(\|x\|) \le a^n + \|x\| - 1 \le a^n + \left(\frac{r}{2^{n+9}}\right)^n \le 2 \cdot \left(\frac{r}{2^{n+9}}\right)^n,$$

we obtain

$$\begin{aligned}\|T(x,a)\| &\le \left(2 \cdot \left(\frac{r}{2^{n+9}}\right)^n\right)^2 \|x\| \le 4 \frac{r^{2n}}{(2^{n+9})^{2n}}\left(1 + \left(\frac{r}{2^{n+9}}\right)^n\right) \\ &< \frac{1}{2}\left(r\left(1 - \frac{1}{(2^{n+9})^n}\right)/2^{n+6}\right)^{2n}.\end{aligned}$$

Therefore

$$TB_r(x,a) \supset B_{\frac{1}{2}r^{2n}((1-1/(2^{n+9})^n)/2^{n+6})^{2n}}(T(x,a)). \qquad \blacksquare$$

Remark: One can see that the order of the co-uniformity module $\omega(r)$ at zero varies for different cases: in the first case $\omega(r) \sim r^{5n}$, in the second $\omega(r) \sim r^{2n+1}$ and in the third it is of order r^{2n}.

Proof of Lemma 2: Note that the matrix $\frac{\partial}{\partial \varphi_j} U^{[n+1]}_{\varphi_1,\varphi_2,\ldots,\varphi_n}$ has operator norm not greater than 2^{j-1}, because it is a sum of 2^{j-1} matrices of norm 1. Therefore

$$\|U^{[n+1]}_{\varphi_1,\varphi_2,\ldots,\varphi_n} - U^{[n+1]}_{\widetilde{\varphi}_1,\widetilde{\varphi}_2,\ldots,\widetilde{\varphi}_n}\| \le \sum_{j=1}^{n} 2^{j-1}[(\varphi_j - \widetilde{\varphi}_j) \bmod 2\pi].$$

Hence if $\gamma \in [\frac{1}{k+1}, \frac{1}{k}]$ satisfies (5) below, then for all x such that $\|x\| \le 1$,

$$\|U^{[n+1]}_{2\pi/\gamma, 2\pi/\gamma^2,\ldots,2\pi/\gamma^n} x - U^{[n+1]}_{\varphi_1,\varphi_2,\ldots,\varphi_n} x\| \le \sum_{j=1}^{n-1} 2^{j-1} \frac{4\pi}{k} \le \frac{2^{n+1}\pi}{k}. \qquad \blacksquare$$

LEMMA 3: *For any $\varphi_1, \varphi_2, \ldots, \varphi_n \in [0, 2\pi]$ and any positive integer $k \ge 2$ there exists $\gamma \in [\frac{1}{k+1}, \frac{1}{k}]$ such that*

$$(5) \quad \varphi_j - \frac{2\pi}{\gamma^j} \bmod 2\pi \le \frac{4\pi}{k} \quad \textit{for all } j = 1, \ldots, n-1 \quad \text{and} \quad \varphi_n - \frac{2\pi}{\gamma^n} \bmod 2\pi = 0.$$

Proof: Let $N(j) = (k+1)^j - k^j - 1$. We define the sequence $\{a_m^{[n]}\}_{m=0}^{N(n)}$ by

$$a_m^{[n]} = 2\pi(k^n + m) + \varphi_n.$$

Now for each $j = n, \ldots, 2$, having constructed the sequence $\{a_m^{[j]}\}_{m=0}^{N(j)}$ such that

$$a_m^{[j]} \in [2\pi(k^j + m), 2\pi(k^j + m + 1)] \quad \text{and} \quad a_m^{[j]} - \varphi_j \bmod 2\pi \le 4\pi/k$$

we construct $\{a_m^{[j-1]}\}_{m=0}^{N(j-1)}$ as follows. Note first that the derivative of the function $q_j(t) = 2\pi(t/2\pi)^{(j-1)/j}$ is less than $1/k$ for $t \in [2\pi k^j, 2\pi(k+1)^j]$. This implies that

$$q_j(a_0^{[j]}) - q_j(2\pi k^j) \le (a_0^{[j]} - 2\pi k^j)\frac{1}{k} \le \frac{2\pi}{k}$$

and, for $0 \le m \le N(j) - 1$,

$$q_j(a_{m+1}^{[j]}) - q_j(a_m^{[j]}) \le (a_{m+1}^{[j]} - a_m^{[j]})\frac{1}{k} \le \frac{4\pi}{k}.$$

Also

$$q_j(2\pi(k+1)^j) - q_j(a_{N(j)}^{[j]}) \le (2\pi(k+1)^j - a_{N(j)}^{[j]})\frac{1}{k} \le \frac{2\pi}{k}.$$

It follows that we can choose $\{a_m^{[j-1]}\}_{m=0}^{N(j-1)}$ among $\{q_j(a_m^{[j]})\}_{m=0}^{N(j)}$ so that

$$a_m^{[j-1]} \in [2\pi(k^{j-1} + m), 2\pi(k^{j-1} + m + 1)] \quad \text{and} \quad a_m^{[j-1]} - \varphi_{j-1} \bmod 2\pi \le 4\pi/k.$$

Consider $\{a_m^{[j]}\}_{m=0}^{N(j)}$ for $j = 1$ — this is one point. Let us define $\gamma = 2\pi/a_0^{[1]}$. Then $2\pi/\gamma^j$ belongs to $\{a_m^{[j]}\}_{m=0}^{N(j)}$ for each j, so (5) holds. ∎

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